A Computer Method for the Dynamic Analysis of a System of Rigid Bodies in Plane Motion

Hazem Ali Attia*

Depertment of Mathematics, College of Science, King Saud University (Al-Qasseem Branch), P.O.Box 237, Buraidah 81999, KSA

This paper presents a computer method for the dynamic analysis of a system of rigid bodies in plane motion. The formulation rests upon the idea of replacing a rigid body by a dynamically equivalent constrained system of particles. Newton's second law is applied to study the motion of the resulting system of particles without introducing any rotational coordinates. A velocity transformation is used to transform the equations of motion to a reduced set. For an open-chain, this process automatically eliminates all of the non-working constraint forces and leads to an efficient integration of the equations of motion. For a closed-chain, suitable joints should be cut and few cut-joints constraint equations should be included. An example of a closed-chain is used to demonstrate the generality and efficiency of the proposed method.

Key Words : Dynamic Analysis, Mechanisms, System of Rigid Bodies, Matrix Formulation, Recursive Formulation

1. Introduction

In recent years, there have been many attempts to develop efficient methods for generating the equations of motion for multibody systems (Schiehlen, 1990; 1997; Shabana, 1994; 1997). These methods can be divided into two main approaches depending on the type of coordinates chosen. In the first approach, the equations of motion are formulated using relative joint coordinates (Keat and Turner, 1984; Paul and Krajcinovic, 1970; Wittenburg, 1977). For an open-chain all the generalized coordinates are independent, while for a closed-chain a minimum number of dependent coordinates should be defined. This leads to an efficient solution and integration of the equations of motion. However, in many applications, this approach leads to a relatively complex recursive formulation based on the loop closure equations and it is also difficult to incorporate general constraints and forcing functions.

The second approach uses absolute coordinates that describe the location and orientation of the bodies in the system with respect to an inertial reference frame (Orlandea et al., 1977; Nikravesh, 1988). In this approach each body has identical coordinate representation and the constraint equations are easily formulated for each joint. A main disadvantage of this method is the large number of mixed differential and algebraic equations which leads to inefficient solution and integration of the equations of motion. Another approach for multibody dynamics modelling uses both coordinate types (Kim and Vanderploeg, 1986; Nikravesh and Gim, 1989). The equations of motion are first formulated in terms of the absolute coordinates and then transformed to relative joint coordinates through the use of a velocity transformation matrix.

One elegant method for generating the equations of motion for multibody systems has been presented in several papers by Garcia de Jalon et

* **E-mail** : ah1113@yahoo.com
TEL : +966-6-3800319; **FAX** : +966-6-3800911-9312
Depertment of Mathematics, College of Science, King Saud University (Al-Qasseem Branch), P.O.Box 237, Buraidah 81999, KSA. (Manuscript **Received** February 18, 2003; **Revised** December 8, 2003)

al.(1982 ; 1986). This method takes advantage of a rudimentary idea of describing a body as a collection of points and vectors. The idea may initially appear as a step backward in the evolutionary process of generating the equations of motion. However, the method exhibits many interesting and extremely useful features. The coordinates and components of points and vectors that are defined to describe a body are dependent on each other through kinemtaic constriants. For example, we may define twelve coordinates and six constraints to describe a free body in space. Furthermore, additional constraints are introduced to represent the kinematic joints interconnecting the rigid bodies. This process yields a large set of loosely coupled differential-algebraic equations of motion. However, these equations can be converted to a minimal or a small set, as in the body-joint coordinates formulation.

In this paper, we first present some of the ideas that appear in (Garcia de Jalon et al., 1982 ; 1986) and a few other papers by the same authors, with a different slant. Although the general ideas are adopted from those references, the methodology of deriving of the equations, and many of the techniques presented in this paper are new. Here, the bodies are described only by points. The mass and the external forces associated with each point are determined, respectively, as a function of the inertial characteristics of the body and the applied forces acting on the body. The equations of motion are derived using the equations of motion for a system of particles and the Lagrange multipliers. Then, the equations of motion are transformed to a reduced set in terms of a selected set of relative joint variables using a velocity transformation matrix which allows efficient solution and integration of the equations of motion without loss of generality. The dynamic analysis of a closed chain is chosen to demonstrate the efficiency and generality of the formulation.

2. Construction of the Equivalent System of Particles

A system of three particles is chosen to replace a rigid body in plane motion as shown in Fig. 1.

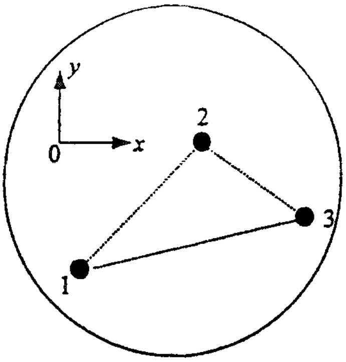

Fig. 1 The rigid body with its equivalent system of three particles

The rigid body and its dynamically equivalent system of particles should have the same mass, the same position of the centre of mass and the same polar moment of inertia about an axis perpendicular to the plane of motion. These conditions are expressed as

$$m=\sum_{i=1}^{3} m_i \tag{1a}$$

$$m\bar{\mathbf{r}}_G=\sum_{i=1}^{3} m_i\bar{\mathbf{r}}_i \tag{1b}$$

$$I_o=\sum_{i=1}^{3} m_i\bar{\mathbf{r}}_i^T\bar{\mathbf{r}}_i \tag{1c}$$

where m is the mass of the rigid body, $\bar{\mathbf{r}}_G$ is the position vector of the centre of mass of the body with respect to a body attached coordinate frame, I_o is the moment of inertia of the rigid body about an axis perpendicular to the plane of motion, m_i is the mass of particle i and $\bar{\mathbf{r}}_i$ is the position vector of particle i of the equivalent system with respect to the body-fixed coordinate frame. Equation (1) represents a system of 4 algebraic equations in 9 unknowns. Five unknowns may be chosen as free variables and then Eq. (1) is solved for the remaining unknowns. The mass of particle 2 and the Cartesian coordinates of particles 1 and 2 with respect to the reference frame are taken as free variables. Especially for $\bar{\mathbf{r}}_1=0$, masses m_1 and m_3 as well as the Cartesian coordinates of particle 3 can be estimated from Eq. (1) in the following closed form :

$$m_3 = \frac{(m\bar{\mathbf{r}}_G - m_2\bar{\mathbf{r}}_2)^T (m\bar{\mathbf{r}}_G - m_2\bar{\mathbf{r}}_2)}{I_o - m_2\bar{\mathbf{r}}_2^T\bar{\mathbf{r}}_2}$$

$$m_1 = m - m_2 - m_3$$

$$\bar{\mathbf{r}}_3 = \frac{m\bar{\mathbf{r}}_G - m_2\bar{\mathbf{r}}_2}{m_3}$$

The positions of mass 1 and 2 may be chosen conveniently, e.g. in case of a rod they may be located at both ends of the rod in order to describe the position of the joints. The other remaining free variable m_2 should be chosen such that,

$$m_2\bar{\mathbf{r}}_2 \neq m\mathbf{r}_G$$

$$m_2\bar{\mathbf{r}}_2^T\bar{\mathbf{r}}_2 \neq I_o$$

In the case of a rigid rod of length l and mass m, particles 1 and 2 are located arbitrarely at both ends of the rod while particle 3 is located at the middle of the rod. The equality conditions for the mass, position vector of the centre of mass and moment of inertia can be solved to determine the unknown masses of the particles in the form,

$$m_2 = \frac{4}{l^2}(mll_G - I_o)$$

$$m_3 = \frac{2}{l^2}\left(I_o - \frac{m}{2}\,ll_G\right)$$

$$m_1 = m - m_2 - m_3$$

where l_G is the location of the centre of mass of the rod with respect to the position of particle 1 and I_o is the polar moment of inertia about an axis perpendicular to the rod and passing through its end associated with particle 1.

If the rigid body is connected to other bodies in a serial chain by revolute joints, then particles 1 and 2 can be conveniently located at the centers of these joints. Two adjacent rigid bodies contribute to the mass concentrated at the joint connecting them. This process reduces the total number of particles replacing the whole system and leads to the automatic elimination of the constraint forces associated with the revolute joints connecting the bodies.

3. Distribution of Forces and Couples

Figure 2 shows a rigid body in plane motion with its equivalent system of particles. The positions of the three particles are given with respect to a body attached coordinate frame. The rigid body is acted upon by an external force with known magnitude, direction, and the location of its point of influence P with respect to the body coordinate frame. The resulting forces and their moments about any arbitrary point should sum up to the original force and its moment about the same point respectively. These conditions are expressed as,

$$\sum_{i=1}^{3}\mathbf{f}_i = \mathbf{f},\ \sum_{i=1}^{3}\bar{\mathbf{r}}_i x \mathbf{f}_i = \bar{\mathbf{r}}_P x \mathbf{f} \qquad (2)$$

where $\mathbf{f}_i$ is the force acting on particle i, $\mathbf{f}$ is the external applied force, $\bar{\mathbf{r}}_i$ is the position vector of particle i with respect to the body-fixed coordinate frame, and $\bar{\mathbf{r}}_P$ is the position vector of the attachment point of the external force with respect to the body-fixed coordinate frame. Distributing the external force over the three particles yields,

$$\mathbf{f}_i = a_i\mathbf{f},\ i = 1, \cdots, 3 \qquad (3)$$

where a_i is the distribution parameter that determines the magnitude of the force affecting on particle i. Solving Eqs. (2) and (3) for a_i gives,

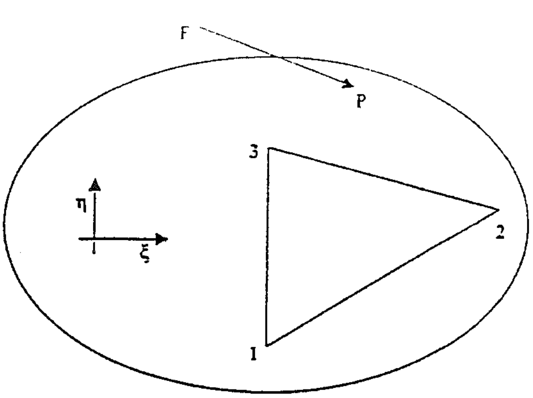

Fig. 2 Planar rigid body with an external force applied

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \xi_2 & \xi_3 \\ 0 & \eta_2 & \eta_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \xi_P \\ \eta_P \end{bmatrix}$$

In the case of a rod only two parameters are required, since the local coordinate frame can be located with the ξ-axis coinciding with the line joining the two particles resulting in $\eta_2=\eta_3=\eta_P=0$ and thus

$$\begin{bmatrix} 1 & 1 \\ 0 & \xi_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \xi_P \end{bmatrix}$$

Also applied force couples may be replaced analytically by an equivalent system of forces distributed over the particles. The resultant of the forces should vanish and the summation of their moments should sum up to the moment of the external applied couple. The applied force couple is replaced by two forces acting on the particles. The magnitude of the force is determined as, $f=m_c/d$ where m_c is the magnitude of the couple and d is the distance between the two particles.

It should be pointed out that the impact and friction forces between bodies can also be modelled in terms of the resulting system of particles by arbitrarly locating the equivalent particles (or introducing additional particles) at the known point of application of these forces. Consequently, impact and friction forces will be acted upon the chosen particles which will be taken into consideration while driving the equations of motion for these particles.

4. Equations of Motion of a Rigid Body

The rigid body shown in Fig. 1 is replaced with a dynamically equivalent constrained system of three particles. The equations of motion of the resulting system of particles are derived by applying Newton's second law to study the translational motion of the particles in the form,

$$m_i\ddot{\mathbf{r}}_i+\mathbf{c}_i=\mathbf{f}_i,\ i=1, \cdots, 3 \tag{4}$$

where m_i is the mass of particle i, $\ddot{\mathbf{r}}_i$ is the acceleration vector of particle i, $\mathbf{f}_i$ is the resultant of the external forces acting on particle i, and $\mathbf{c}_i$ is the resultant of the constraint forces acting on particle i. The geometric constraint equations take the form,

$$(\mathbf{r}_1-\mathbf{r}_2)^T(\mathbf{r}_1-\mathbf{r}_2)-d_{1,2}^2=0 \tag{5a}$$

$$(\mathbf{r}_1-\mathbf{r}_3)^T(\mathbf{r}_1-\mathbf{r}_3)-d_{1,3}^2=0 \tag{5b}$$

$$(\mathbf{r}_2-\mathbf{r}_3)^T(\mathbf{r}_2-\mathbf{r}_3)-d_{2,3}^2=0 \tag{5c}$$

where $d_{i,j}$ is the distance between particles i and j. Equation (5) is written to fix the distance between particles 1 and 2, 1 and 3, and 2 and 3, respectively. By expressing the constraint forces in terms of Lagrange multipliers, the equations of motion can be written in the following final form,

$$\begin{bmatrix} \mathrm{M} & -\Phi_q^T \\ \Phi_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \gamma \end{bmatrix} \tag{6}$$

where $\ddot{\mathbf{r}}=[\ddot{\mathbf{r}}_1^T, \ddot{\mathbf{r}}_2^T, \ddot{\mathbf{r}}_3^T]^T$ is the unknown acceleration vector of the particles, $\mathbf{f}=[\mathbf{f}_1^T, \mathbf{f}_2^T, \mathbf{f}_3^T]^T$ is the vector of external forces distributed over the boundary particles, ϕ_r is the Jacobian matrix of the constraint Eq. (5), $\mathbf{r}$ is the vector of Cartesian coordinates of the particles, $\lambda=[\lambda_1, \lambda_2, \lambda_3]^T$ are the Lagrange multipliers associated with the constraint Eq. (5), γ is the right hand side of the acceleration equations of the constraint Eq. (5), $\gamma=-2(\dot{\mathbf{r}}_i-\dot{\mathbf{r}}_j)^T(\dot{\mathbf{r}}_i-\dot{\mathbf{r}}_j)$ for the distance constraint between particles i and j, and $\mathbf{M}$ is the 6×6 diagonal mass matrix given by,

$$\mathbf{M}=\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_3 \end{bmatrix}$$

Equation (6) and the constraint Eq. (5) represent a system of 9 differential-algebraic equations that can be solved for the unknown accelerations of the three particles and the Lagrange multipliers.

Similar procedures can be followed to obtain the equations of motion of a rigid rod where the constraint equations between the particles are given by

$$(\mathbf{r}_1-\mathbf{r}_2)^T(\mathbf{r}_1-\mathbf{r}_2)-d_{1,2}^2=0 \tag{7a}$$

$$\mathbf{r}_3-(\mathbf{r}_1+\mathbf{r}_2)/2=\mathbf{0} \tag{7b}$$

By utilizing Eq. (7b) to eliminate the acceleration of the middle particle 3, we obtain the equations of motion in the form given by Eq. (6) where $\ddot{\mathbf{r}}=[\ddot{\mathbf{r}}_1^T, \ddot{\mathbf{r}}_2^T]^T$ is the unknown acceleration vector of the boundary particles, $\mathbf{f}=[\mathbf{f}_1^T, \mathbf{f}_2^T]^T$ is the vector of external forces distributed over the particles, ϕ_r is the Jacobian matrix of the distance constraint Eq. (7a), $\lambda=\lambda_1$ is the Lagrange multiplier associated with the constraint Eq. (7a), $\gamma=-2(\dot{\mathbf{r}}_1-\dot{\mathbf{r}}_2)^T(\dot{\mathbf{r}}_1-\dot{\mathbf{r}}_2)$, and $\mathbf{M}$ is the 4×4 symmetric sparse mass matrix given by,

$$\mathbf{M}=\begin{bmatrix} M_1 & 0 & m_3/4 & 0 \\ 0 & M_1 & 0 & m_3/4 \\ m_3/4 & 0 & M_2 & 0 \\ 0 & m_3/4 & 0 & M_2 \end{bmatrix}$$

and where $M_1=m_1+m_3/4$, $M_2=m_2+m_3/4$.

5. Equations of Motion of a System of Rigid Bodies

A multibody system is a collection of rigid bodies interconnected by kinematic joints and/or force elements. If there are no kinematic joints in the system, it is called a system of unconstrained bodies. If there are one or more kinematic joints in the system, it is referred to as a system of constrained bodies. In the case of unconstrained system, the rigid bodies do not share any common particles. Then, the overall mass matrix is constructed by assembling the different mass matrices for the individual bodies in a block-diagonal matrix $\mathbf{M}$. The final form of the equations of motion is given by,

$$\begin{bmatrix} \mathbf{M} & -\mathbf{D}^{(g)T} \\ \mathbf{D}^{(g)} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \lambda^{(g)} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \gamma^{(g)} \end{bmatrix} \tag{8}$$

where $\mathbf{M}$ is the overall mass matrix, $\ddot{\mathbf{r}}$ are the acceleration vectors of the particles, $\mathbf{D}^{(g)}$ is the Jacobian matrix of the geometric constraints between the particles, $\lambda^{(g)}$ are the Lagrange multipliers associated with the geometric constraints between the particles, $\mathbf{g}$ is the resultant vector of external forces acting on the particles, $\gamma^{(g)}$ is the right hand side of the acceleration equations of the geometric constraints, and (g) is a superscript standing for geometric constraints.

In the case of constrained system, two or more adjacent bodies usually share one or more particles. The mass matrices corresponding to these bodies are overlapped. The overall mass matrix is constructed by assembling the individual mass matrices and adding the contributions of the common particles. Due to the kinematic joints, additional kinematic constraints exist. The final form of the equations of motion for a general constrained system is given by;

$$\begin{bmatrix} \mathbf{M} & -\mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \gamma \end{bmatrix} \tag{9}$$

$\mathbf{D}$ is the Jacobian matrix of the constraints between the particles and of joints, λ are the Lagrange multipliers associated with the constraints between the particles and of joints, and γ is the right hand side of the acceleration equations of the constraints. The matrix form of the resulting equations of motion indicates that the formulation is suitable for numerical computation and computer implementation.

Therefore, all kinds of mechanisms, that consist of interconnected rigid bodies with kinematic joints, can be dealt with using the suggested algorithm. All the rigid bodies are replaced with a constrained system of equivalent particles. Some kinematic joint constraints can be eliminated by properly locating the particles (e.g. revolute joint) while other joint constraints must be formulated in terms of the Cartesian coordinates of the particles.

6. Equations of Motion in the Joint Coordinates

Derivation of the equations of motion in terms of the equivalent system of particles is simple and straightforward. The main disadvantage of this formulation is the assignment of a large number of dependent generalized coordinates which results in a large number of coupled differential-algebraic equations. The numerical solution and integration of this mixed set of equations are computationally inefficient. The equations of motion, written initially in terms of the Cartesian coordinates of the particles, are transformed to a reduced set of equations in terms of a selected set of relative joint variables. This transformation is done using a velocity transformation matrix which relates the Cartesian velocities of the particles to the relative joint velocities and allows an efficient solution and integration of the equations of motion.

6.1 Equations of motion for an open loop system

For a multibody system with open loops the equations of motion in terms of the Cartesian coordinates of the particles as given by Eq. (9), are transformed to a minimal set of differential equations equal to the number of degrees of freedom of the system. In the process of transformation, the position of a body is defined with respect to its adjacent reference body by relative angles or distances. Therefore, the vector of joint coordinates is determined by the type of the kinematic joints. The vectors of joint coordinates, velocities and accelerations are defined by,

$$\theta=[\theta_1, \cdots, \theta_n]^T$$
$$\dot{\theta}=[\dot{\theta}_1, \cdots, \dot{\theta}_n]^T$$
$$\ddot{\theta}=[\ddot{\theta}_1, \cdots, \ddot{\theta}_n]^T$$

where n is the number of degrees of freedom of the system. The Cartesian velocities of the particles are related to the joint velocities as,

$$\dot{\mathbf{r}}=\mathbf{B}\ \dot{\theta} \tag{10}$$

where the matrix $\mathbf{B}$ is the velocity transformation matrix which is a function of the Cartesian coordinates of the particles and depends on the topology of the system as well as on the type of the kinematic joints connecting the adjacent bodies. The time derivative of Eq. (10) gives the acceleration transformation as

$$\ddot{\mathbf{q}}=\mathbf{B}\ \ddot{\theta}+\dot{\mathbf{B}}\ \dot{\theta} \tag{11}$$

Substituting the Cartesian accelerations from Eq. (11) in the first of Eq. (9) and premultiplying by the matrix $\mathbf{B}^T$ yields,

$$\mathbf{B}^T\mathbf{MB}\ \ddot{\theta}-\mathbf{B}^T\mathbf{D}^T\ \lambda=\mathbf{B}^T(\mathbf{g}-\mathbf{M}\dot{\mathbf{B}}\ \dot{\theta}) \tag{12}$$

The second term in Eq. (12) vanishes and the final form of the equations of motion in terms of the relative joint variables, if all the joint variables are independent, is given as ;

$$\bar{\mathbf{M}}\ \ddot{\theta}=\mathbf{f} \tag{13}$$

where $\bar{\mathbf{M}}=\mathbf{B}^T\mathbf{MB}$ and $\mathbf{f}=\mathbf{B}^T(\mathbf{g}-\mathbf{M}\dot{\mathbf{B}}\ \dot{\theta})$. Solving the symmetric linear system of Eq. (13) yields the unknown joint accelerations. The estimated joint accelerations are integrated for given initial joint coordinates and velocities to determine the updated values of the joint coordinates and velocities.

6.2 Equations of motion for a closed loop system

For multibody systems containing closed kinematic loops additional constraint equations are introduced due to cut-joints. Every closed loop is cut at one of the kinematic joints in order to produce a reduced open loop system. For this reduced system, joint variables are defined for every open loop. The resulting equations of motion in this case are a coupled set of differential-algebraic equations. Let the cut-joint constraint equations for closed kinematic loops be expressed as ;

$$\psi(\theta)=\mathbf{0} \tag{14}$$

The first and second time derivatives of the constraints are given by ;

$$\dot{\psi}=\mathbf{C}\ \dot{\theta}=\mathbf{0} \tag{15}$$

$$\ddot{\psi}=\mathbf{C}\ \ddot{\theta}+\dot{\mathbf{C}}\ \dot{\theta}=\mathbf{0} \tag{16}$$

where $\mathbf{C}$ is the Jacobian matrix of the cut-joint constraints. The constraint forces associated with the cut-joints constraints are expressed in terms of Lagrange multipliers and introduced into the equations of motion for open loop system Eq. (13),

$$\begin{bmatrix}\bar{\mathbf{M}} & -\mathbf{C}^T\\ \mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ v\end{bmatrix}=\begin{bmatrix}\mathbf{f}\\ -\dot{\mathbf{C}}\ \dot{\theta}\end{bmatrix} \tag{17}$$

where v is the vector of Lagrange multipliers associated with the cut-joint constraints. Equations (14) to (17) represent the equations of motion for a multibody system when the number of selected joint coordinates is greater than the number of degrees of freedom of the system.

It should be noted that in this formulation, the kinematic constraints due to some common types of kinematic joints (e.g. revolute or spherical joints) can be automatically eliminated by properly locating the equivalent particles. The remaining kinematic constraints along with the geometric constraints are, in general, either linear or quadratic in the Cartesian coordinates of the particles. Therefore, the coefficients of their Jacobian matrix are constants or linear in the rectangular Cartesian coordinates, whereas in the formulation based on the relative coordinates the constraint equations are derived from loop closure equations which have the disadvantage that they do not directly determine the positions of the links and points of interest which makes the establishment of the dynamic problem more difficult. Also, the resulting constraint equations are highly nonlinear and contain complex circular functions. The absence of these circular functions in the point coordinate formulation leads to faster convergence and better accuracy. Furthermore, preprocessing the mechanism by the topological graph theory is not necessary as it would be the case with loop constraints.

Also, in comparison with the absolute coordinate formulation, the manual work of the local axes attachment and local coordinates evaluation as well as the use of the rotational variables and the rotation matrices in the absolute coordinate formulation are not required in the point coordinate formulation. This leads to fully computerized analysis and accounts for a reduction in the computational time and memory storage. In addition to that, the constraint equations take much simpler forms as compared with the absolute coordinates. Furthermore, the use of absolute coordinates may cause numerical problems if differences of large values of the absolute coordinates are used, e.g. for the calculation of spring or damper forces or constraint residuals.

The elimination of the rotational coordinates, in the presented formulation, leads to possible savings in computation time when this procedure is compared against the absolute or relative coordinate formulation. It has been determined that numerical computations associated with rotational transformation matrices and their corresponding coordinate transformations between reference frames is time consuming and, therefore, if these computations are avoided more efficient codes may be developed. The elimination of rotational coordinates can also be found very beneficial in design sensitivity analysis of multibody systems. In most procedures for design sensitivity analysis, leading to an optimal design process, the derivatives of certain functions with respect to a set of design parameters are required. Analytical evaluation of these derivatives are much simpler if the rotational coordinates are not present and if we only deal with translational coordinates.

Some practical applications of multibody dynamics require one or more bodies in the system to be described as deformable in order to obtain a more realistic dynamic response. Deformable bodies are normally modeled by the finite element technique. Assume that the deformable body is connected to a rigid body described by a set of particles. Then, one or more particles of the rigid body can coincide with one or more nodes of the deformable body in order to describe the kinematic joint between the two bodies. This is a much simpler process than when the rigid body is described by a set of translational and rotational coordinates. In general, the point coordinates have additional advantages over the other systems of coordinates since they are the most suitable coordinates for the graphics routines and the animation programs.

6.3 Integration of the equations of motion

The differential equations of motion for an open-chain, Eq. (13), or for a closed-chain, Eq. (17), represent a set of nonlinear ordinary differential equations with the time as an independent variable. The system can be put in the standard form $\dot{\mathbf{y}}=\mathbf{f}(\mathbf{y}, \mathbf{t})$, where $\mathbf{y}$ and $\dot{\mathbf{y}}$ are the vectors that contain the relative joint coordinates, velocities, and accelerations as,

$$\dot{\mathbf{y}}=\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} \quad \text{and} \quad \mathbf{y}=\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix}$$

The numerical solution of the equations of motion requires a numerical integration process that determines the elements of $\mathbf{y}$ at every time step. The function $\mathbf{f}$ is evaluated by solving the equations of motion for the unknown joint accelerations. This numerical process is summarized as follows:

(1) Initially, the joint coordinates and velocities are known, i.e. $\mathbf{y}_0=[\theta_0^T, \dot{\theta}_0^T]^T$.

(2) Using the vector $\mathbf{y}_0=[\theta_0^T, \dot{\theta}_0^T]^T$ the Cartesian coordinates and velocities and hence the matrix $\mathbf{B}$, $\dot{\mathbf{B}}$, $\mathbf{C}$ and $\dot{\mathbf{C}}$ can be constructed.

(3) With the knowledge of the known constant mass matrix $\mathbf{M}$ and the force vector $\mathbf{f}$ the equations of motion for open-chain, Eq. (13), or a closed-chain, Eq. (17), can be derived.

(4) Solve the equations of motion for $\ddot{\theta}$ and v using the Gaussian elimination technique adopted for symmetric matrices.

(5) Construct the $\dot{\mathbf{y}}$ vector and return the contents to the integration algorithm.

(6) Repeat the previous steps at every time step.

Gear's method (Gear, 1988) for the numerical integration of differential-algebraic equations is used to overcome the instability problem resulting during the modelling process of constrained mechanical systems. Use of both Cartesian and joint coordinates produces an efficient set of equations without loss of generality.

7. Dynamic Analysis of a Closed-Chain

The mechanism shown in Fig. 3(a) represents an example of a closed-chain. It is an one degree of freedom mechanism which has two independent closed loops OABC and CBDEF. The rigid body ABD is replaced by three particles; 2, 3, and 4 as shown in Fig. 3(b). The other rigid links are replaced by three particles: two particles are located at both ends while the third one is located at the middle of the rod. The adjacent bodies share common particles, as indicated in Fig. 3(b). The mass and moment of inertia of each body is given in Table 1 while the masses of the equivalent particles are given in Table 2. A resultant 14×14 sparse mass matrix is constructed. Since two closed loops exist, two joints should be cut.

Table 1 Description of the rigid bodies

Body #	Mass (kg)	Inertia (kg·m^2)
1	0.5	0.0266
2	3.0	6.0000
3	1.0	0.4408
4	1.0	1.0208
5	1.0	1.0208

Table 2 Masses (kg) of the equivalent particles

Body 1	0.0833	0.0833	0.3333
Body 2	4.2967	1	−2.2967
Body 3	0.1666	0.1666	0.6666
Body 4	0.1666	0.1666	0.6666
Body 5	0.1666	0.1666	0.6666

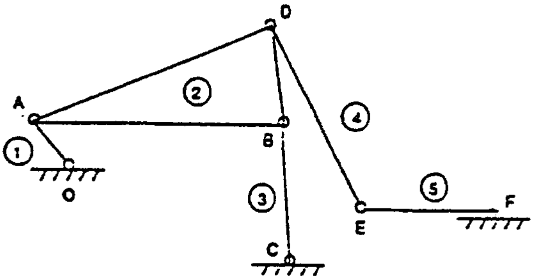

Fig. 3(a) Schematic view of a mechanism

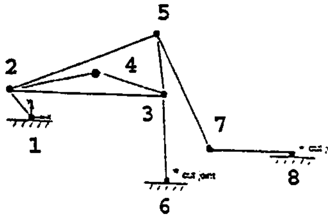

Fig. 3(b) Schematic diagram of the mechanism with the replacing particles

Cutting the revolute joints at C and F produces the two open loops OABC and OADEF. For the first open loop, the vector of joint coordinates is defined by $[\theta_1, \theta_{2,1}, \theta_{3,2}]^T$, where θ_1 is the inclination angle of body 1 with the horizontal, $\theta_{2,1}$ and $\theta_{3,2}$ are the relative angles between bodies 2 and 1 and bodies 3 and 2, respectively. For the second open loop, the vector of joint coordinates is defined by $[\theta_1, \theta_{2,1}, \theta_{4,2}, \theta_{5,4}]^T$, where $\theta_{4,2}$ and $\theta_{5,4}$ are the relative angles between bodies 4 and 2 and bodies 5 and 4, respectively. Then, the final vector of joint coordinates is given by,

$$\theta=[\theta_1,\ \theta_{2,1},\ \theta_{3,2},\ \theta_{4,2},\ \theta_{5,4}]^T$$

The velocity transformation equation has the form,

$$\begin{bmatrix}\dot{\mathbf{r}}_2\\ \dot{\mathbf{r}}_3\\ \dot{\mathbf{r}}_4\\ \dot{\mathbf{r}}_5\\ \dot{\mathbf{r}}_6\\ \dot{\mathbf{r}}_7\\ \dot{\mathbf{r}}_8\end{bmatrix}=\begin{bmatrix}\mathbf{k}x\mathbf{d}_{2,1} & & & & \\ \mathbf{k}x\mathbf{d}_{3,1} & \mathbf{k}x\mathbf{d}_{3,2} & & & \\ \mathbf{k}x\mathbf{d}_{4,1} & \mathbf{k}x\mathbf{d}_{4,2} & & & \\ \mathbf{k}x\mathbf{d}_{5,1} & \mathbf{k}x\mathbf{d}_{5,2} & & & \\ \mathbf{k}x\mathbf{d}_{6,1} & \mathbf{k}x\mathbf{d}_{6,2} & \mathbf{k}x\mathbf{d}_{6,3} & & \\ \mathbf{k}x\mathbf{d}_{7,1} & \mathbf{k}x\mathbf{d}_{7,2} & & \mathbf{k}x\mathbf{d}_{7,5} & \\ \mathbf{k}x\mathbf{d}_{8,1} & \mathbf{k}x\mathbf{d}_{8,2} & & \mathbf{k}x\mathbf{d}_{8,5} & \mathbf{k}x\mathbf{d}_{8,7}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_{2,1}\\ \dot{\theta}_{3,2}\\ \dot{\theta}_{4,2}\\ \dot{\theta}_{5,2}\end{bmatrix}$$

where $\mathbf{k}$ is a unit vector normal to the plane of motion and $d_{i,j}=\mathbf{r}_i-\mathbf{r}_j$. Two constraint equations are written at the cut joints to prevent the motion of particles 6 and 8, in the vector form;

$$\mathbf{r}_6-\mathbf{c}_1=0,\ \mathbf{r}_8-\mathbf{c}_2=0$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are known constant vectors. The corresponding velocity and acceleration equations take the form,

$$\dot{\mathbf{r}}_6=\dot{\mathbf{r}}_8=0,\ \ddot{\mathbf{r}}_6=\ddot{\mathbf{r}}_8=0$$

The 4×5 $\mathbf{C}$ matrix, defined in Eq. (15), can be put in the following closed form,

$$\mathbf{C}=\begin{bmatrix}\mathbf{k}x\mathbf{d}_{6,1} & \mathbf{k}x\mathbf{d}_{6,2} & \mathbf{k}x\mathbf{d}_{6,3} & 0 & 0\\ \mathbf{k}x\mathbf{d}_{8,1} & \mathbf{k}x\mathbf{d}_{8,2} & 0 & \mathbf{k}x\mathbf{d}_{8,5} & \mathbf{k}x\mathbf{d}_{8,7}\end{bmatrix}$$

The resulting equations of motions, as given by Eq. (17), represent a 9×9 symmetric linear system that can be solved for the unknown joint accelerations and reaction forces at the cut joints. The motion of the mechanism is controlled by the vertical gravitational forces and constraint forces. For DAP-2D program, which is based on the absolute coordinates [9], a system of 15+14 differential equations of motion plus algebraic equations of constraints is constructed. Thus a resulting system of 29 differential-algebraic equations should be solved at every time step to determine the unknown accelerations and reaction forces. This reduction in the number of differential equations of motion and in turn the number of integrable variables is considered as an advantage of the presented formulation. The motion of the mechanism starts from rest where the initial Cartesian coordinates of the particles are given in Table 3. Figure 4 presents the trajectory of particle 7 in the plane of motion. The results of the simulation are tested and compared with the DAP-2D program which is based on the absolute coordinates formulation [9]. The comparison shows a complete agreement between the two formulations.

Table 3 Initial coordinates of the points

Point O	(0, 0)	Point D	(2.793, 2.737)
Point A	(−0.566, 0.566)	Point E	(4, −0.548)
Point B	(3.029, 0.751)	Point F	(7.5, −0.5)
Point C	(3.5, −1.5)		

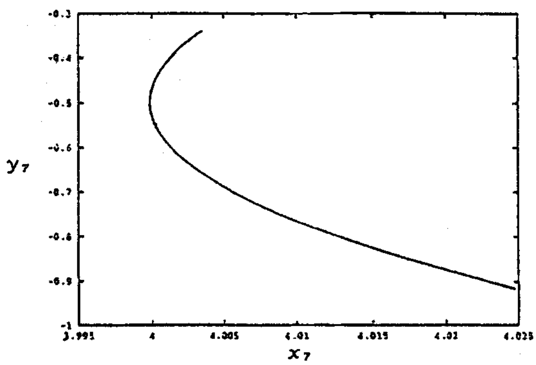

Fig. 4 The trajectory of particle 7

Conclusion

In this paper a computer method for the dynamic analysis of planar mechanisms is developed. Initially a rigid body is replaced by a dynamically equivalent constrained system of particles and the equations of motion are formulated using only Newton's second law in terms of the Cartesian coordinates of the particles. This eliminates the need to use any rotational coordinates or the corresponding transformation matrices which leads to possible savings in computational time. Due to the large number of the defined dependent Cartesian coordinates, which leads to inefficient solution and integration of the equations of motion, the equations of motion are transformed to a reduced set using relative joint coordinates. This leads to an efficient solution and integration of the equations of motion. The formulation is then applied to carry out the dynamic analysis of a closed-chain which incorporates open and closed chains with the common types of kinematic joints. The simulations ensure the generality and efficiency of the formulation.

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